Calculating Partial Inductance of Vias for Printed Circuit Board Modeling

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Abstract

A complete derivation of inductance from energy relations is presented, outlining all the key steps and assumptions. Based on this derivation, the concept of partial inductance is reviewed and several useful expressions for partial inductance are presented. The accuracy of these expressions is then evaluated by comparing these formula to 3D field solutions and measurement data of test structures.

Theoretical Background

Equations for inductance can be derived using anyone of several alternative methods. Here we derive a general formula for inductance from energy relations. From Poynting's Theorem the energy stored in a magnetic field can be written as

$$W_M = \frac{1}{2}\iiint \bar{B} \cdot \bar{H} dV \tag{1}$$

From circuit theory the energy stored in an inductor is given by

$$W_M = \frac{1}{2}Li^2 \tag{2}$$

Equating these two definitions and solving for L gives

$$L = \frac{1}{i^2}\iiint \bar{B} \cdot \bar{H} dV \tag{3}$$

The vector magnetic potential **A** can be defined such that

$$\bar{B} = \nabla \times \bar{A} \tag{4}$$

Substituting (4) into (3) gives

$$L = \frac{1}{i^2}\iiint (\nabla \times \bar{A}) \cdot \bar{H} dV \tag{5}$$

A standard vector identity is

$$(\nabla \times \bar{A}) \cdot \bar{H} = \nabla \cdot (\bar{A} \times \bar{H}) + \bar{A} \cdot (\nabla \times \bar{H}) \tag{6}$$

Using this on equation (5) gives

$$L = \frac{1}{i^2}\iiint \nabla \cdot (\bar{A} \times \bar{H}) dV + \frac{1}{i^2}\iiint \bar{A} \cdot (\nabla \times \bar{H}) dV \tag{7}$$

Applying the divergence theorem and substituting Ampere's law in differential form (under quasi-static conditions) gives

$$L = \frac{1}{i^2}\oint_S (\bar{A} \times \bar{H}) dS + \frac{1}{i^2}\iiint \bar{A} \cdot \bar{J} dV \tag{8}$$

Since the volume integration is over all space, the surface enclosing this volume is at infinity. Applying the boundary condition that the magnetic field intensity **H** must be zero at infinity, the left hand term vanishes. Note that if the current itself were assumed to return at infinity the magnetic field could not be zero at infinity. The volume current density **J** is zero outside of the conductor and therefore the bounds of integration for the right hand integral are over the volume bounded by the conductor. The vector magnetic potential **A** for a volume current is defined as

$$\bar{A} = \frac{\mu}{4\pi}\iiint\frac{\bar{J}}{\bar{R}_{ij}}dV \qquad (9)$$

where $\mathbf{R_{ij}}$ is the distance vector from the line element $d\mathbf{l}$ to the field point. Substituting (9) into (8) yields

$$L_{ij} = \frac{\mu}{4\pi i^2}\int_{v_i}\int_{v_j}\left(\frac{\bar{J}_i}{\bar{R}_{ij}} \cdot \bar{J}_j\right)dV_i dV_j \qquad (10)$$

This is the most general formula for calculating the self and mutual inductance for a given problem definition. Using (10) the inductance can be calculated for two loops, i and j (see Figure 1). If the loops are assumed to have a uniform current density $\mathbf{J_i}$ and $\mathbf{J_j}$, and constant cross-sectional area, a_i and a_j, the following substitutions can be made $dV_i = dl_i dS_i$ and $J_i = i_i / a_i$. Equation (10) can then be rewritten as

$$L_{ij} = \frac{1}{a_i a_j}\frac{\mu}{4\pi}\oint_i\int_{a_i}\oint_j\int_{a_j}\frac{\overline{dl_i} \cdot \overline{dl_j}}{\bar{R}_{ij}}da_i da_j \qquad (11)$$

For loops that are thin filaments, the current density disappears for places off the line contour and Neumann’s formula is obtained

$$L_{ij} = \frac{\mu}{4\pi}\oint_i\oint_j\frac{\overline{dl_i} \cdot \overline{dl_j}}{\bar{R}_{ij}} \qquad (12)$$

By rewriting the integrations over the lengths as summations over segments of the loop the partial inductance concept is introduced [1]

$$L_{ij} = \sum_{k=1}^{K}\sum_{m=1}^{M}\frac{\mu}{4\pi}\oint_{b_k}^{c_k}\oint_{b_m}^{c_m}\frac{\overline{dl_k} \cdot \overline{dl_m}}{\bar{R}_{km}} \qquad (13)$$

where loop i is divided into K segments and loop j is divided into M segments. The line integral is then calculated for each segment over the geometry of each segment in free space. This integration will be well defined for a particular problem definition and produces a *unique* partial inductance value for a given segment geometry.

Closed-Form Expressions for Partial Inductance of Vias

Closed-form expressions for the partial inductance can be obtained from Neumann’s formula. Rosa derived an expression for the partial mutual inductance between two thin filaments of length *h* and separation *d* from Neumann’s formula [2]. The geometrical relation of the filaments is shown in Figure 2. Inserting the problem definition into (12), the integral to be solved is

$$L_{ij} = \frac{\mu}{4\pi}\int_0^h\int_0^h\frac{\overline{dy} \cdot \overline{dy'}}{\sqrt{(y-b)^2 + d^2}} \qquad (14)$$

Solving the double integration in (14) produces

$$L_{ij} = 5.08h\left[\ln\left(\left(\frac{h}{d}\right) + \sqrt{\left(1 + \left(\frac{h}{d}\right)^2\right)}\right) - \sqrt{\left(1 + \left(\frac{d}{h}\right)^2\right)} + \left(\frac{d}{h}\right)\right] \qquad (15)$$

where the units of (15) are in pH and the dimensions are in mils. The partial *self* inductance can be calculated using (15) with the separation of the filaments equal to the conductor radius. If the filament length is much greater than the filament separation, (15) can be shown to reduce to the following expression for partial inductance

$$L_{ij} = 5.08h\left[\ln\left(\left(\frac{2h}{d}\right) - 1\right)\right] \quad (16)$$

Figure 3 plots the partial self inductance calculated using (15) and (16) for a via that is 10 mils in diameter as a function of length. These results are compared to 3D field solution using Ansoft Q3D. As expected, (16) produces non-physical results (negative inductance values) when the aspect ratio of the conductor approach unity. Excellent agreement is obtained using (15) over a wide range of conductor aspect ratios. The largest discrepancy is observed for short conductors when the port definitions in the field solver becomes increasingly important in order to obtain accurate partial self inductance values.

Via Modeling Considerations

In general, accurate modeling of vias for PCB applications requires that the partial self and partial mutual inductances are included for all the vias. However accurate, there is a huge computational penalty to this approach. For example, 10 vias would require 55 elements. The number of elements can be significantly reduced by capturing the partial self inductances only. An approach to assessing the inaccuracy introduced by this simplification was presented in [3].

Via Inductance Measurement Considerations

When attempts are made to measure the inductance of a single, stand-alone via, the measuring instruments/probes create a closed loop around the via (Figure 4). To reduce the contribution of this uncertainty, a pair of vias can be connected in a loop by a plane. Through calibration, the portion of loop inductance created by the finite probe pitch is removed. As long as the coupling among the four sides of this loop is negligible, and assuming that the horizontal plane-connection inductance can be neglected compared to the via inductances, the measured loop inductance will be a good approximation of the sum of the two partial self inductances of vias.

Measurement Results

Figure 5 shows the uncertainty associated with the removal of the probe-tip inductance. The graph shows the impedance reading after a full two-port calibration on adjacent tracks of GGB calibration substrate (20-mil distance between tracks), on the same tracks (labeled 20-mil) and tracks with 150-mil separation. The 20-mil separation yields an approximately 4 pH inductance. Figure 6 shows the measured impedance and extracted inductance of two 90-mil long 22-mil diameter vias with 50-mil center-to-center separation. The partial inductance obtained from 3D field solution and equation (15) are within 20% of the measured inductance.

References

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3. J. Miller and I. Novak, *Modeling of Plane Discontinuities for Power Distribution Networks*, Proceedings of the Progress in Electromagnetic Research Symposium (PIERS), Cambridge, MA (2002).

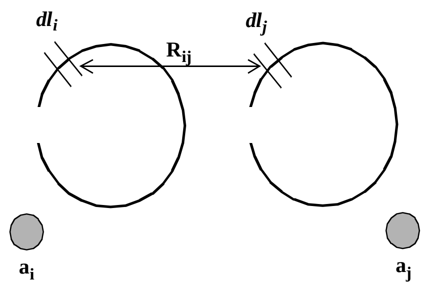

Fig. 1: Two coupled loops, i and j.

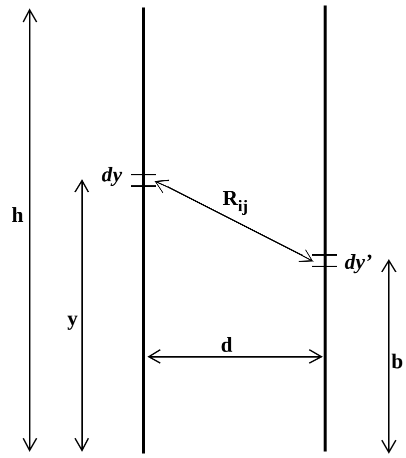

Fig. 2: Two filamentary conductors.

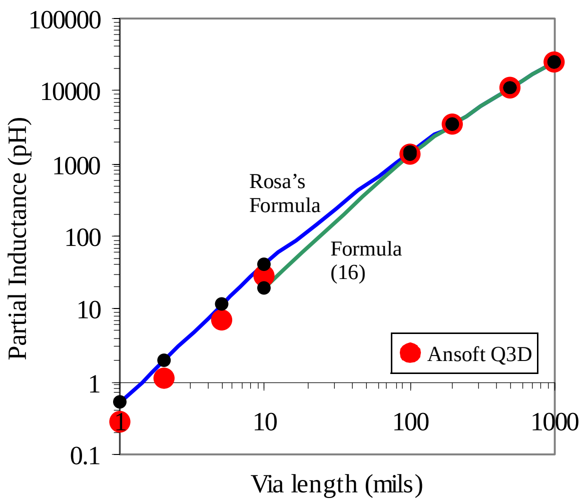

Fig. 3: Via partial inductance as a function of length.

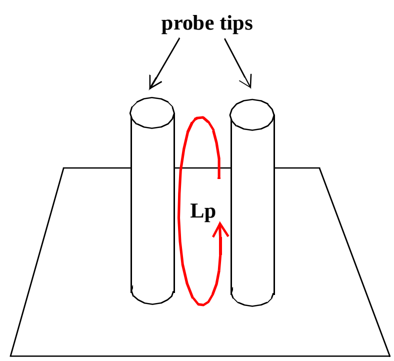

Fig. 4: Closed loop created by probe.

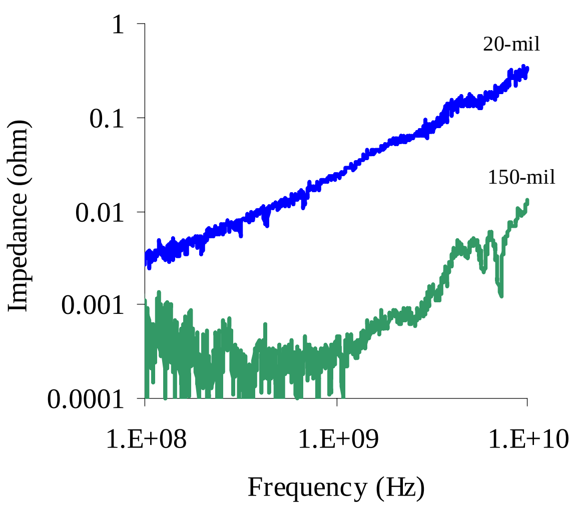

Fig. 5: Probe calibration reading on shorts.

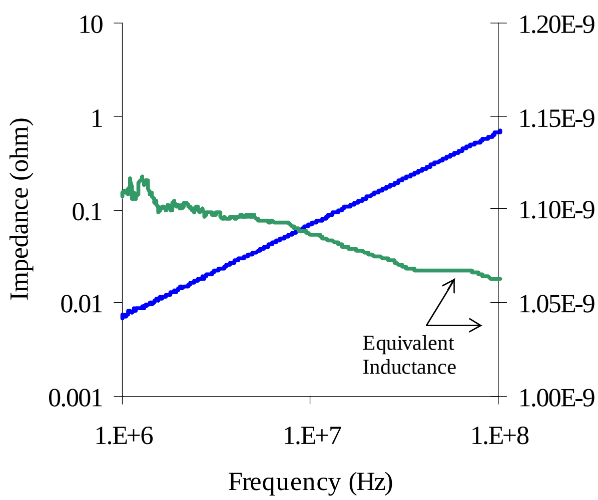

Fig. 6: Measured impedance magnitude and equivalent inductance of a via pair shorted by a solid plane